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Correlation functions and fluctuation–dissipation relation in driven phase ordering systems: an exactly solvable model

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Abstract

The dynamics of a phase ordering system with non-conserved order parameter under a plain shear flow with rate γ is solved analytically in the large- N limit. A phase transition is observed at a critical temperature $T_c(\gamma)$. After a quench from a high temperature equilibrium state to a lower temperature T a non-equilibrium stationary state is entered when $T > T_c(\gamma)$, while aging dynamics characterizes the phases with $T \leq T_c(\gamma)$. Two-time quantities are computed and the off-equilibrium generalization of the fluctuation–dissipation theorem is provided.

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1. Introduction

In the field of modern statistical mechanics many efforts have been devoted to understanding the properties of non-equilibrium. These studies were promoted by the recognition that many relevant systems, such as glasses and coarsening systems, are intrinsically far from equilibrium and, therefore, cannot be described by theories formulated as extensions of Gibbs statistical mechanics. Instead, specific concepts relative to the non-equilibrium state must be developed. A promising subject in this field is the attempt to build an out of equilibrium fluctuation–dissipation theorem (FDT). Specifically, the aim of these studies is to investigate whether the relation between the (integrated) response function χ and the autocorrelation function D is meaningful and bears some relevant information regarding the non-equilibrium state. There are several reasons for expecting this, the most natural being induction from the equilibrium case. In equilibrium these two quantities are linearly related by means of the standard FDT

and a fundamental quantity, the temperature of the system, appears as the coefficient of this relation. For off-equilibrium systems we do not have nowadays an analogous theorem, of such a wide generality, but conjectures, indications or, sometimes, proofs exist claiming that, in restricted contexts, the relation between χ and D can be inferred on general grounds. In particular, the slope of the curve $\chi(D)$ has been interpreted as the *effective* temperature of the system, different from that of the thermal bath.

These concepts, which were originally developed in the study of turbulence [1], have been successively applied to the glassy state [2, 3] and, more generally, to aging systems [4] where the non-equilibrium state is generally determined by the change of a thermodynamic control parameter, as in a temperature quench.

A different non-equilibrium situation is very often considered because of its physical relevance: the case of driven systems. These systems are subjected to an external forcing that prevents them attaining equilibrium. Examples of this situation range from systems subjected to a force which does not derive from a potential, such as sheared fluids or conductor wires with a potential difference applied at their ends, to systems with externally imposed thermal gradients or magnetic materials under the action of an oscillatory magnetic field and tapped materials. Despite many studies on this subject [5] much less is known about the effect of the drive on the fluctuation–dissipation relation, although some progress has been made recently [3, 6].

The complexity of the glassy state and the difficulty of both analytical and numerical results in this field make the task of understanding the mechanisms whereby the FDT can be generalized a hard question. Therefore, in this paper we focus on phase ordering systems which, besides being physically interesting in their own right, can be regarded as simple paradigms of aging behaviour. We solve exactly the dynamics of a model quenched from an initial disordered state under the action of an external driving field. This kind of process is relevant in many physical contexts, as will be discussed later. We consider a coarse-grained system described by the usual double-well φ^4 Hamiltonian where φ is a scalar order parameter field. A Langevin dynamics is considered and the order parameter is not conserved. The driving field is a planar shear or Couette flow. This model, without the shear flow, is referred to in the literature as model A. The choice of this particular flow, besides analytical tractability, is also motivated by its practical relevance, which appears clearly in the physics of complex fluids and binary mixtures, where the behaviour of the stress as a function of the shear rate is one of the main characterizations of fluid systems. Moreover, the structure of binary and complex fluids is strongly affected by the flow and this is relevant in many applications [7] and interesting for its theoretical implications. In the phase separation of binary mixtures, for instance, it was found that not only the morphology but also the laws of dynamical scaling are affected by the shear-induced anisotropy [8, 9] and new phenomena related to the presence of stress in segregating systems, such as logarithmic-time periodic oscillations [10] and violation of dynamical scaling [11], have been observed.

In this paper we focus on a non-conserved order parameter; however we believe that our study could also be useful as a preliminary step for understanding more complex systems with conserved ordered parameter. On its own the model of this paper can describe the behaviour of liquid crystals undergoing the isotropic–nematic transition where the order parameter is not conserved [12]. The phase separation properties of this model have been studied in [13] where only the behaviour of one-time correlations was considered.

The role of the non-linear term in the φ^4 Hamiltonian is crucial for quenching below and at the critical temperature. However its presence makes the analytical study of the model impossible unless one considers approximate theories. Among these the so-called large- N limit has played a prominent role in the study of phase separation. The approximation takes

into account a vectorial system with an infinite number of components N . This model, which can be solved by a self-consistent closure, is one of the few cases where it is possible to illustrate the out-of-equilibrium behaviour of a coarsening system by exact calculations [12].

In the case without shear the large- N approximation captures the essence of the phenomenon at a semi-quantitative level [14]. In this work we use this approximation to calculate the two-time correlation function showing how the dynamical exponents are affected by the presence of the flow. The response function can also be exactly computed allowing a careful discussion of the behaviour of the fluctuation–dissipation relation.

The plan of the paper is the following. In section 2 the model is defined and in section 3 its stationary critical properties are discussed. In section 4 the self-consistent condition is explicitly worked out and one-time quantities are computed. Two-time quantities, the autocorrelation function and the response function, are studied in sections 5 and 6. The violation of the FDT is considered in section 7 and a final discussion is presented in section 8. Appendices A–D, containing some mathematical details, complete the paper.

2. The model

We consider a system with dynamics described by the equations

$$\frac{\partial \varphi^\alpha(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot (\varphi^\alpha(\vec{x}, t) \vec{v}) = -\Gamma \frac{\delta \mathcal{H}[\vec{\varphi}]}{\delta \varphi^\alpha(\vec{x}, t)} + \eta^\alpha(\vec{x}, t) \quad \alpha = 1, \dots, N \quad (1)$$

where $\{\varphi^\alpha\}$ are the N components of the vectorial order parameter. For instance, in the case of magnetic systems, $\vec{\varphi}$ is the local magnetization, Γ is a transport coefficient and the Gaussian white noise η , representing thermal fluctuations, has values

$$\langle \eta^\alpha(\vec{x}, t) \rangle = 0 \quad \langle \eta^\alpha(\vec{x}, t) \eta^\beta(\vec{x}', t') \rangle = 2T \delta_{\alpha\beta} \Gamma \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (2)$$

where T is the temperature of the thermal bath. Without the convective term on the left-hand side, equation (1) would be the usual Langevin equation which governs the purely relaxational dynamics of a system with non-conserved order parameter and Hamiltonian $\mathcal{H}[\vec{\varphi}]$. In that case relations (2) would assure that in thermodynamic equilibrium at temperature T the fluctuation–dissipation theorem is verified. Here equations (2) are supposed to hold on the basis of local equilibrium [5, 15]. More precisely, it is assumed that the non-equilibrium system can be subdivided into cells small enough that any thermodynamic property—which in non-equilibrium situations may depend on space—varies slightly over one cell, but large enough that equilibrium statistical mechanics holds within each cell [16]. Furthermore, since thermodynamic properties out of equilibrium may depend on time, the characteristic time over which a macroscopic fluctuation dies away within one cell must be much smaller than the typical evolution time of the system. This implies that over intervals much smaller than this evolution time local equilibrium is maintained in each cell and the fluctuation–dissipation theorem holds. This will be recovered in section 7.

The velocity \vec{v} in equation (1) is chosen to be the steady planar shear flow

$$\vec{v} = \gamma y \vec{e}_x \quad (3)$$

where γ is the spatially homogeneous shear rate and \vec{e}_x is a unit vector in the flow direction. We observe that, in spite of the presence of a velocity field proportional to a space coordinate, translational invariance still holds, since a shift of a in the y -direction is equivalent to a Galilean transformation to a new reference frame globally moving with an added velocity $a\gamma$ in the x -direction. This allows us in the following to introduce Fourier transforms and the standard

definition of the structure factor [8, 15]. The Hamiltonian has the Ginzburg–Landau form

$$\mathcal{H}[\vec{\varphi}] = \int_V d^d x \left[\frac{1}{2} |\nabla \vec{\varphi}|^2 + \frac{r}{2} \vec{\varphi}^2 + \frac{g}{4N} (\vec{\varphi}^2)^2 \right] \quad (4)$$

where $r < 0$, $g > 0$ and V is the volume of the system. We will be interested in processes where the system is initially in the equilibrium ($\gamma = 0$) infinite temperature state with expectations

$$\begin{cases} \langle \varphi^\alpha(\vec{x}, 0) \rangle = 0 \\ \langle \varphi^\alpha(\vec{x}, 0) \varphi^\beta(\vec{x}', 0) \rangle = \Delta_0 \delta_{\alpha\beta} \delta(\vec{x} - \vec{x}') \end{cases} \quad (5)$$

and then is suddenly put in contact with a heat bath at temperature T , and subjected to the shear flow. Δ_0 is a constant.

In the large- N limit [12] the equation of motion for the Fourier transform $\varphi^\alpha(\vec{k}, t) = \int_V d^d x \varphi^\alpha(\vec{x}, t) \exp(i\vec{k} \cdot \vec{x})$ takes the linear form

$$\frac{\partial \varphi^\alpha(\vec{k}, t)}{\partial t} - \gamma k_x \frac{\partial \varphi^\alpha(\vec{k}, t)}{\partial k_y} = -\Gamma[k^2 + I(t)]\varphi^\alpha(\vec{k}, t) + \eta^\alpha(\vec{k}, t) \quad (6)$$

where

$$\langle \eta^\alpha(\vec{k}, t) \rangle = 0 \quad \langle \eta^\alpha(\vec{k}, t) \eta^\beta(\vec{k}', t') \rangle = 2T \delta_{\alpha\beta} \Gamma V \delta_{\vec{k}, -\vec{k}'} \delta(t - t') \quad (7)$$

and the function

$$I(t) = r + \frac{g}{N} \langle \vec{\varphi}^2(\vec{x}, t) \rangle \quad (8)$$

has to be calculated self-consistently. The quantity $\langle \vec{\varphi}^2(\vec{x}, t) \rangle$ can be expressed as

$$\frac{1}{N} \langle \vec{\varphi}^2(\vec{x}, t) \rangle = \frac{1}{V} \sum_{\vec{k}} C(\vec{k}, t) \quad (9)$$

where the structure factor

$$C(\vec{k}, t) = \frac{1}{NV} \langle \vec{\varphi}(\vec{k}, t) \cdot \vec{\varphi}(-\vec{k}, t) \rangle \quad (10)$$

is the solution of the equation

$$\frac{\partial C(\vec{k}, t)}{\partial t} - \gamma k_x \frac{\partial C(\vec{k}, t)}{\partial k_y} = -2\Gamma[k^2 + I(t)]C(\vec{k}, t) + 2\Gamma T. \quad (11)$$

The momentum sum in equation (9) extends up to a phenomenological ultraviolet cut-off Λ .

3. Stationary properties

Letting $\partial C(\vec{k}, t)/\partial t = 0$ in equation (11), and setting $\Gamma = 1$, the stationary form of the structure factor reads [15]

$$C(\vec{k}) = 2T \int_0^\infty e^{-2 \int_0^z \mathcal{K}^2(t') dt' - 2z\xi_\perp^{-2}} dz \quad (12)$$

where

$$\vec{\mathcal{K}}(s) = \vec{k} + \gamma s k_x \vec{e}_y \quad (13)$$

and

$$\xi_\perp^{-2} = \lim_{t \rightarrow \infty} I(t) = r + \frac{g}{V} \sum_{\vec{k}} C(\vec{k}). \quad (14)$$

This quantity plays the role of a *transverse* correlation length, because the modes of $C(\vec{k})$ with $k_x = 0$ take the usual Ornstein–Zernike form

$$C(k_x = 0, \vec{k}_\perp) = \frac{T}{k_\perp^2 + \xi_\perp^{-2}}. \quad (15)$$

Inserting equation (12) into equation (14) one has

$$\xi_\perp^{-2} = r + \frac{g}{V} 2T \sum_{\vec{k}} \int_0^\infty e^{-2 \int_0^\xi \kappa^2(t') dt' - 2z\xi_\perp^{-2}} dz. \quad (16)$$

This equation can be conveniently solved separating the $\vec{k} = 0$ term under the sum. Then, for very large volumes

$$\xi_\perp^{-2} = r + gTP(\xi_\perp^{-2}) + g \frac{T}{V\xi_\perp^{-2}} \quad (17)$$

where

$$P(\xi_\perp^{-2}) = 2 \int \frac{d^d k}{(2\pi)^d} e^{-k^2/\Lambda^2} \int_0^\infty e^{-2 \int_0^\xi \kappa^2(t') dt' - 2z\xi_\perp^{-2}} dz. \quad (18)$$

The existence of a microscopic lengthscale proportional to Λ^{-1} corresponds to the presence of a microscopic relaxation time

$$\tau_M = (2\Lambda^2)^{-1} \quad (19)$$

which has to be compared with the shear flow timescale

$$\tau_s = \gamma^{-1}. \quad (20)$$

As usual, it is assumed [8, 16] that the flow does not distort the structure of the system at the microscopic level so that $\tau_M \ll \tau_s$ and

$$A \equiv \frac{2\Lambda^2}{\gamma} \gg 1. \quad (21)$$

In the following we will concentrate on the particular cases $d = 2$ and $d = 3$. The function $P(x)$ is a non-negative monotonically decreasing function with the maximum value at $x = 0$. It is related through equation (71) to the function $f^L(x)$ calculated in appendix A and, from equations (79) and (86), its value at $x = 0$ is given by

$$P(0) = \begin{cases} \frac{1}{4\pi} (\ln(\sqrt{12}A) + \ln 2) & d = 2 \\ \frac{1}{2(2\pi)^{3/2}} \left(\sqrt{\gamma A} - \Gamma^2(3/4) \frac{\sqrt{\gamma}}{\sqrt{2\pi} 3^{1/4}} \right) & d = 3 \end{cases} \quad (22)$$

where $\Gamma(3/4)$ is the Gamma function evaluated at $3/4$. By graphical analysis one can easily show that equation (17) admits a solution with a finite value of ξ_\perp^{-2} for all T . However, there exists a critical value of the temperature $T_c(\gamma)$ defined by

$$r + gT_c(\gamma)P(0) = 0 \quad (23)$$

such that for $T > T_c(\gamma)$ the solution is independent of the volume, while for $T \leq T_c(\gamma)$ it does depend on V . From equations (22) and (23) one has

$$T_c(\gamma) = \begin{cases} (4\pi M_0^2) / \ln(A) & d = 2 \\ T_c(\gamma = 0) [1 - \Gamma^2(3/4) 3^{-1/4} \sqrt{2\pi/A}]^{-1} & d = 3 \end{cases} \quad (24)$$

with $M_0^2 = -r/g$ and $T_c(\gamma = 0) = 4M_0^2\pi^{3/2}/\Lambda$. Note that the effect of the flow is to increase the value of the critical temperature with respect to the case with $\gamma = 0$ [17], according to

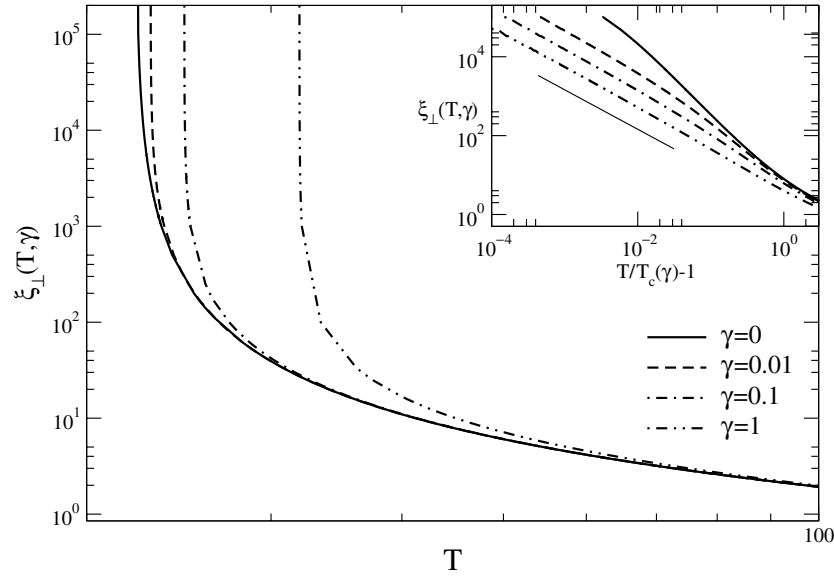


Figure 1. ξ_{\perp} is shown against T for different values of γ in $d = 3$ with $\Lambda = 1$, $-r = g = 1$. The inset shows the same function plotted against $[T - T_c(\gamma)]/T_c(\gamma)$. The reference line below the data curves represents the expected slope $1/2$ (equation (25)). γ is measured in units of Γ .

$T_c(\gamma) \sim 1/|\ln \gamma|$ in $d = 2$ and $T_c(\gamma) - T_c(0) \propto \sqrt{\gamma}$ in $d = 3$. In $d = 2$ this produces a qualitative difference because a phase transition occurs at a finite temperature, differently from the unsheared case.

Finally, concerning the behaviour of the transverse correlation length, above $T_c(\gamma)$ with $[T - T_c(\gamma)]/T_c(\gamma) \ll 1$, from equations (17) and (18) and the expression of f^L calculated in appendix A (equations (78) and (85)), one finds

$$\xi_{\perp}^{-2} = M_0^2 \left(\frac{T - T_c(\gamma)}{T_c(\gamma)} \right) (T A_3 + 1/g)^{-1} \quad d = 3 \quad (25)$$

$$\xi_{\perp}^{-2} |\log(\xi_{\perp}^{-2})| = \frac{M_0^2}{T B_2} \left(\frac{T - T_c(\gamma)}{T_c(\gamma)} \right) \quad d = 2 \quad (26)$$

where A_3 and B_2 are defined in appendix A. Hence, for the exponent ν defined by $\xi_{\perp} \sim \left(\frac{T - T_c(\gamma)}{T_c(\gamma)} \right)^{-\nu}$ one finds $\nu = 1/2$, with logarithmic corrections for $d = 2$. At $T_c(\gamma)$, one has $\xi_{\perp}^{-2} \sim 1/\sqrt{V}$ in $d = 3$ and $\xi_{\perp}^{-2} |\log(\xi_{\perp}^{-2})| \sim 1/V$ for $d = 2$. The behaviour of ξ_{\perp} for $T \geq T_c(\gamma)$ is shown in figure 1.

For fixed γ , ξ_{\perp} decreases by increasing the temperature. Indeed this is expected because coherence is suppressed by thermal fluctuations. The role of γ is more subtle. Naively one would expect that increasing γ produces a smaller ξ_{\perp} , because coherent islands are washed away by the flow. However it must also be considered that γ raises the critical temperature $T_c(\gamma)$. Hence, for fixed T , larger values of γ make $[T - T_c(\gamma)]/T_c(\gamma)$ smaller, the system gets closer to the critical point and the coherence length is increased. This second effect competes with the former and produces a net increase of ξ_{\perp} with γ . However, by changing γ and T in order to maintain a constant distance from criticality, namely keeping $[T - T_c(\gamma)]/T_c(\gamma)$ fixed, the second effect is suppressed and one sees that, indeed, ξ_{\perp} is reduced by the flow (see the inset of figure 1). For small $[T - T_c(\gamma)]/T_c(\gamma)$ the power law behaviour (25) is observed.

Finally, for large temperatures, ξ_{\perp} becomes γ -independent, because the decoherence produced by thermal fluctuations prevails over shear effects.

Finally, turning to the phase $T < T_c(\gamma)$, the transverse correlation length is given by $\xi_{\perp}^{-2} = T/(M^2V)$ where

$$M = M_0 \sqrt{1 - \frac{T}{T_c(\gamma)}} \tag{27}$$

is the analogue of the spontaneous magnetization in equilibrium.

4. Dynamical solution

We will solve the model at asymptotic times in the case of quenching processes with initial conditions given by (5). From now on we will take the infinite volume limit. The formal solution of equation (11) is given by

$$C(\vec{k}, t) = \frac{\Delta_0}{y(t)} e^{-2\int_0^t du \mathcal{K}^2(u)} e^{-k^2/\Lambda^2} + \frac{2T}{y(t)} \int_0^t e^{-2\int_0^{t-z} du \mathcal{K}^2(u)} e^{-k^2/\Lambda^2} y(z) dz \tag{28}$$

where

$$y(t) = \exp\left(2 \int_0^t [r + gS(t')] dt'\right) \tag{29}$$

and, from equation (9)

$$S(t) = \lim_{V \rightarrow \infty} \frac{1}{N} \langle \vec{\varphi}^2(\vec{x}, t) \rangle = \lim_{V \rightarrow \infty} \frac{1}{V} \sum_{\vec{k}} C(\vec{k}, t) = \int \frac{d^d k}{(2\pi)^d} e^{-k^2/\Lambda^2} C(\vec{k}, t). \tag{30}$$

Then, defining $f(t)$ as

$$f(t) \equiv \int \frac{d^d k}{(2\pi)^d} e^{-2\int_0^t [\mathcal{K}^2(\varepsilon)] dz - k^2/\Lambda^2} \tag{31}$$

$$= (8\pi)^{-d/2} (t + 1/2\Lambda^2)^{-d/2} 2 \left(4 - \frac{\gamma^2 t^4}{(t + 1/(2\Lambda^2))^2} + \frac{4}{3} \frac{\gamma^2 t^3}{t + 1/(2\Lambda^2)} \right)^{-1/2} \tag{32}$$

one has

$$S(t) = \frac{1}{y(t)} \left(\Delta_0 f(t) + 2T \int_0^t f(t - t') y(t') dt' \right). \tag{33}$$

We observe that the correction induced by the flow in equation (32) is independent of the dimensionality of the system and that, in the absence of flow, the expected behaviour $f(t) \approx (8\pi t)^{-d/2}$ is recovered.

From equations (29) and (33), one obtains the integro-differential equation

$$\frac{dy(t)}{dt} = 2y(t)(r + gS(t)) = 2y(t)r + 2g \left(\Delta_0 f(t) + 2T \int_0^t f(t - t') y(t') dt' \right) \tag{34}$$

that can be solved introducing Laplace transforms denoted by $f^L(s) = \int_0^\infty f(t) e^{-st} dt$ and assuming that $y^L(s)$, the Laplace transform of $y(t)$, is well defined, as will be verified *a posteriori*.

Equation (34) implies that $y^L(s)$ can be expressed in terms of $f^L(s)$ through the equation

$$y^L(s) = \frac{1 + 2\Delta_0 f^L(s)}{s - 2r - 4gT f^L(s)}. \tag{35}$$

The function $f^L(s)$ has been calculated in appendix A. Then, from equation (35), inverting the Laplace transform (see appendix B for details), the function $y(t)$ can be obtained. For $t \gg \tau_s$, $y(t)$ behaves as

1. $d = 2$

$$y(t) = \begin{cases} \frac{1}{2B_2T} \left(\frac{1}{2} + \frac{M_0^2 \Delta_0}{2T_c(\gamma)} \right) \frac{1}{\log(2\xi_{\perp}^{-2})} e^{-2\xi_{\perp}^{-2}t} & T > T_c(\gamma) \\ \frac{1}{2B_2T_c} \left(\frac{1}{2} + \frac{M_0^2 \Delta_0}{2T_c(\gamma)} \right) \frac{1}{\log t} & T = T_c(\gamma) \\ \frac{B_2}{M^4} (T + \Delta_0 M_0^2) t^{-2} & T < T_c(\gamma) \end{cases} \quad (36)$$

2. $d = 3$

$$y(t) = \begin{cases} \frac{1}{2} e^{-2\xi_{\perp}^{-2}t} \frac{1 + M_0^2 \Delta_0 / T_c(\gamma)}{1 + 4T_c(\gamma)A_3} & T > T_c(\gamma) \\ \frac{1}{2} \frac{1 + M_0^2 \Delta_0 / T_c}{1 + 4T_c(\gamma)A_3} & T = T_c(\gamma) \\ \frac{3}{4} \frac{1}{\sqrt{\pi}} \frac{B_3}{M^4} (T + \Delta_0 M_0^2) t^{-5/2} & T < T_c(\gamma) \end{cases} \quad (37)$$

where M is given in equation (27), ξ_{\perp} and $T_c(\gamma)$ are given in equations (24)–(26) and the parameters A_3 , B_2 , B_3 are defined in appendix A. Equations (36) and (37) hold for $t > \xi_{\perp}^2$ in the high temperature phase $T > T_c(\gamma)$ and for $t \gg \tau_s$ when $T \leq T_c(\gamma)$.

5. Autocorrelation function

In this section we analyse the asymptotic behaviour of the autocorrelation function

$$\begin{aligned} D(t, t') &\equiv \langle \varphi(\vec{r}, t) \varphi(\vec{r}, t') \rangle \\ &= \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \mathcal{D}(\vec{k}, t; \vec{k}', t') e^{-k^2/2\Lambda^2} e^{-k'^2/2\Lambda^2} \end{aligned} \quad (38)$$

where the correlator

$$\mathcal{D}(\vec{k}, t; \vec{k}', t') = \langle \varphi(\vec{k}, t) \varphi(\vec{k}', t') \rangle \quad (39)$$

satisfies the equation

$$\frac{\partial \mathcal{D}(\vec{k}, t; \vec{k}', t')}{\partial t} - \gamma k_x \frac{\partial \mathcal{D}(\vec{k}, t; \vec{k}', t')}{\partial k_y} = -[k^2 + gS(t) + r] \mathcal{D}(\vec{k}, t; \vec{k}', t') \quad (40)$$

with initial condition $\mathcal{D}(\vec{k}, t'; \vec{k}', t') = C(\vec{k}', t') \delta(\vec{k} + \vec{k}')$ and $t \geq t'$. The function $D(t, t')$ is calculated in appendix D using equations (36) and (37) for $y(t)$. Here we discuss the results in the three cases $T > T_c(\gamma)$, $T = T_c(\gamma)$ and $T < T_c(\gamma)$.

1. $T > T_c(\gamma)$. In the regime $t' > \xi_{\perp}^2$ equations (116) and (118) show that the correlation function becomes the time-translation invariant (TTI) quantity

$$\begin{aligned} D(t, t') &\simeq D_{\text{st}}(\tau, \xi_{\perp}) \\ &= \frac{4T}{(8\pi)^{d/2}} \int_{\tau/2}^{\infty} e^{-2\xi_{\perp}^{-2}y} (y + 1/2\Lambda^2)^{-d/2} \frac{1}{\sqrt{4 + \frac{1}{3}\gamma^2 y^2 + \frac{\gamma^2}{2}\tau^2(1 - \frac{\tau^2}{8y^2})}} dy \end{aligned} \quad (41)$$

with $\tau = t - t'$. This implies that ξ_{\perp}^2 is the characteristic relaxation time of fluctuations above the critical temperature.

2. $T = T_c(\gamma)$. At the critical temperature, due to equations (119) and (121), $D(t, t')$ tends, for $t' \gg \tau_s$ (equation (20)), to the form $D_{st}(\tau, \infty)$, namely

$$D(t, t') \simeq D_{st}(\tau, \infty) = \frac{4T}{(8\pi)^{d/2}} \int_{\tau/2}^{\infty} (y + 1/2\Lambda^2)^{-d/2} \frac{1}{\sqrt{4 + \frac{1}{3}\gamma^2 y^2 + \frac{\gamma^2}{2}\tau^2(1 - \frac{\tau^2}{8y^2})}} dy. \quad (42)$$

Then one has TTI again but, for large τ , $D_{st}(\tau, \infty)$ decays as a power law $D_{st}(\tau, \infty) \propto \tau^{-d/2}$, implying the absence of a characteristic relaxation time, as usual at criticality. Note that the shear has the effect of increasing the exponent of the power law decay with respect to the value $-(d - 2)/2$ of the undriven case. In other terms, the flow tends to decorrelate faster the system.

3. $T < T_c(\gamma)$. In this case there are two time regimes of interest:

- (i) short time separation: $t' \rightarrow \infty$, $\frac{t'}{t} \rightarrow 0$ (quasi-stationary regime)
- (ii) large time separation: $t' \rightarrow \infty$, $\frac{t'}{t} \rightarrow \infty$ (aging regime).

In the time sector (i), from equations (125) one has

$$D(t, t') = M^2 + D_{st}(\tau, \infty) \quad (43)$$

where $D_{st}(\tau, \infty)$ is the same time translational invariant quantity found at $T = T_c(\gamma)$, equation (42).

On the other hand, in the limit (ii), from equation (126), one gets

$$D(t, t') = D_{ag}(t/t') = M^2 \left(\frac{t'}{t}\right)^{\frac{d+2}{4}} (1 + t'/t)^{-\frac{d+2}{2}} \frac{2^{\frac{d+2}{2}}}{\sqrt{4\frac{2-(1-t'/t)^3}{1+t'/t} - 3\frac{(2-(1-t'/t)^2)^2}{(1+t'/t)^2}}}. \quad (44)$$

Note the dependence of D_{ag} on the ratio t/t' alone, as usual in slowly relaxing aging systems. Furthermore, for $t/t' \gg 1$, a generalization of the Fisher–Huse exponent λ defined by

$$D(t, t') \propto (t'/t)^\lambda \quad (45)$$

gives

$$\lambda = \frac{d+2}{4}. \quad (46)$$

The structure of the autocorrelation function provided in equations (44) and (43) for large t' allows its splitting into the sum of two distinct contributions

$$D(t, t') = D_{st}(\tau, \infty) + D_{ag}(t/t'). \quad (47)$$

In the regime (i) $D_{ag}(t/t')$ remains fixed at its initial value $D_{ag}(1) = M^2$, while $D_{st}(\tau, \infty)$ decays to zero. This makes equation (47) consistent with equation (43). Conversely, in the regime (ii), $D_{st}(\tau, \infty)$ has already decayed to zero and the whole time dependence is carried by $D_{ag}(t/t')$, providing again consistency between equations (47) and (44).

This whole pattern of behaviour of $D(t, t')$ is closely reminiscent of what is known for the system without shear [18]. In that case it was shown that these features reflect an underlying structure of the order parameter field which, in the late stage of the dynamics of a quench below T_c , can be decomposed into two statistically independent stochastic components, $\varphi(\vec{x}, t) = \psi(\vec{x}, t) + \sigma(\vec{x}, t)$, responsible for the stationary and aging parts of the autocorrelation function respectively, as discussed in [19]. In the present case, by proceeding along the lines

of the case $\gamma = 0$, it is straightforward to show that the same splitting of the order parameter field can be explicitly exhibited, which accounts for the structure of $D(t, t')$ discussed above; we refer to [18, 20] for further details. In systems with a scalar order parameter [21] the physical interpretation of this decomposition is the following: given a configuration at a time well inside the late stage scaling regime, one can distinguish degrees of freedom $\psi(\vec{x}, t)$ in the bulk of domains from those $\sigma(\vec{x}, t)$ pertaining to the interfaces. The first ones are driven by thermal fluctuations and behave locally as in equilibrium; the second ones retain memory of the noisy initial condition, and produce the aging behaviour. For large N , where topological defects are unstable and domains are not well defined, the recognition of the physical degrees of freedom associated with $\psi(\vec{x}, t)$ and $\sigma(\vec{x}, t)$ is not straightforward as for scalar systems. Nevertheless, the possibility of also splitting the order parameter in this case indicates that the same fundamental property is shared by systems with different N .

The overall behaviour of the autocorrelation function is shown in figure 2, for different t' . This figure also shows a comparison with the case $\gamma = 0$. The pattern is qualitatively similar in both cases: initially a fast decay to a plateau value is observed. This is due to the vanishing of $D_{\text{st}}(\tau, \infty)$ in the quasi-stationary regime while $D_{\text{ag}}(t/t')$ remains constant, $D_{\text{ag}}(t/t') \simeq M^2$ (this value corresponds to the height of the plateau in the figure). Note that curves with different t' collapse in this regime, due to the stationarity of $D_{\text{st}}(\tau, \infty)$. For longer times, when $t - t' \sim t'$, the autocorrelation function departs from the plateau value, because also $D_{\text{ag}}(t/t')$ starts to fall off, and a pronounced dependence on the waiting time t' , or aging, is observed. After the stationary regime is over (around $\tau \sim 10$) γ also plays an important role. Actually one observes that the decay of $D_{\text{ag}}(t/t')$ is promoted by the shear, as expected because correlations are *washed out* by the flow. Indeed, considering the case with $t' = 10^5$, for example, $D(t, t')$ decays from one to the value 10^{-3} in a time τ of order 10^9 for $\gamma = 0$ and in a time of order 3×10^7 for $\gamma = 0.1$. For long times this effect results in the larger Fisher–Huse exponent (46) with respect to the undriven case, where $\lambda = d/4$ [12].

6. Linear response

In order to calculate the response of the field to an external perturbation $\vec{h}(\vec{x}, t)$ we add to the original Hamiltonian the term $\delta\mathcal{H} = - \int dx^d \vec{h}(\vec{x}, t) \cdot \vec{\varphi}(\vec{x}, t)$. The Langevin equation (1) for a generic component becomes

$$\frac{\partial \varphi^\alpha(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot (\varphi^\alpha(\vec{x}, t) \vec{v}) = - \frac{\delta \mathcal{H}[\vec{\varphi}]}{\delta \varphi^\alpha(\vec{x}, t)} + h^\alpha(\vec{x}, t) + \eta^\alpha(\vec{x}, t). \quad (48)$$

The two-time response function is defined as

$$\mathcal{R}(\vec{k}, t; \vec{k}', t') = (2\pi)^d \left. \frac{\delta \langle \varphi^\alpha(\vec{k}, t) \rangle}{\delta h^\alpha(-\vec{k}', t')} \right|_{\langle h(\vec{k}', t') \rangle = 0} \quad (49)$$

with $t \geq t'$. Here, due to rotational symmetry, a generic component of the order parameter can be considered and vectorial indices can be dropped. Using the properties $\langle \varphi^\alpha(\vec{k}, 0) \rangle = 0$ and $\langle \eta^\alpha(\vec{k}, t) \rangle = 0$ we get

$$\mathcal{R}(\vec{k}, t; \vec{k}', t') = (2\pi)^d \delta(\vec{\mathcal{K}}(t - t') + \vec{k}') \sqrt{\frac{y(t')}{y(t)}} e^{-\int_0^{t-t'} du \mathcal{K}^2(u)} \quad (50)$$

where the function $y(t)$ is defined in equation (29).

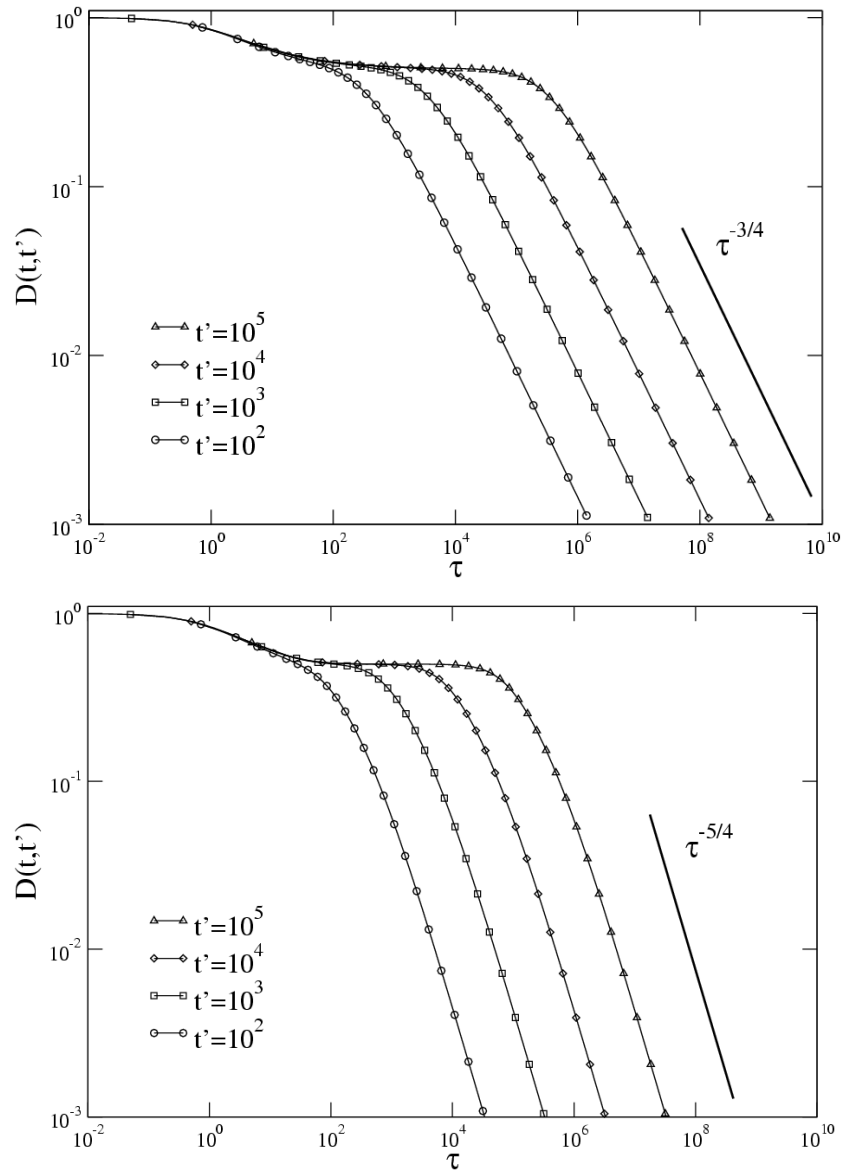


Figure 2. Autocorrelation function versus τ for different waiting times t' , with $T = T_c(\gamma)/2$, $d = 3$, $\Lambda = 1$, $-r = g = 1$. The upper figure refers to the case without shear, the lower one to the case with $\gamma = 0.1$. Times are measured in units of Γ^{-1} , shear rates in units of Γ .

The autoresponse function can also be introduced as

$$\begin{aligned}
 R(t, t') &\equiv \left. \frac{\delta \langle \varphi(\vec{r}, t) \rangle}{\delta h(\vec{r}, t')} \right|_{h=0} \\
 &= \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \mathcal{R}(\vec{k}, t; \vec{k}', t') e^{-k^2/2\Lambda^2} e^{-k'^2/2\Lambda^2}
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{y(t')}{y(t)}} \int \frac{d^d k}{(2\pi)^d} e^{-\int_0^{t-t'} du \mathcal{K}^2(u)} e^{-(k^2 + \frac{d(y')}{2})/\Lambda^2} \\
&= \frac{2^{1-d/2}}{(4\pi)^{d/2}} \sqrt{\frac{y(t')}{y(t)}} \left(\frac{t-t'}{2} + \tau_M\right)^{-d/2} \frac{1}{\sqrt{4 + \frac{4}{3}\gamma^2 \frac{(t-t')^3}{t-t'+2\tau_M} - \gamma^2 \frac{(t-t')^4}{(t-t'+2\tau_M)^2}}} \quad (51)
\end{aligned}$$

where $d(\gamma) = \mathcal{K}^2(t-t') - k^2$.

Real experiments and numerical simulations usually consider the integrated response function, namely the response to a constant perturbation switched on at time t_w and acting until the time t . For instance, in spin systems perturbed by a magnetic field, χ corresponds to the so-called zero-field cooled magnetization [22]. For this quantity one obtains

$$\chi(t, t_w) \equiv \int_{t_w}^t R(t, t') dt' = \frac{2^{1-d/2}}{(4\pi)^{d/2}} \frac{1}{\sqrt{y(t)}} \int_0^\tau dz \sqrt{y(t-z)} \frac{\left(\frac{z}{2} + \tau_M\right)^{-d/2}}{\sqrt{4 + \frac{1}{3}\gamma^2 z^2}} \quad (52)$$

where we have introduced the variables $z = t - t'$, $\tau = t - t_w$ and neglected the two asymptotically irrelevant terms containing τ_M in the square root in the last line of equation (51). We will evaluate this function in the asymptotic regime $t_w \rightarrow \infty$ using the large time behaviour of $y(t)$ given in equations (36) and (37).

1. $T > T_c(\gamma)$. Inserting equations (36) and (37) in equation (52), the response function reads

$$\chi(t, t_w) = \chi_{\text{st}}(\tau, \xi_\perp) = \frac{2}{(4\pi)^{d/2}} \int_0^\tau dz e^{-2\xi_\perp^2 z/2} \frac{(z + 2\tau_M)^{-d/2}}{\sqrt{4 + \frac{1}{3}\gamma^2 z^2}} \quad (53)$$

which is a time translational invariant quantity, as expected, because the system reaches a stationary state.

2. $T = T_c(\gamma)$. For $t_w \gg \tau_s$ one has

$$\chi(t, t_w) = \chi_{\text{st}}(\tau, \infty) = \frac{2}{(4\pi)^{d/2}} \int_0^\tau dz \frac{(z + 2\tau_M)^{-d/2}}{\sqrt{4 + \frac{1}{3}\gamma^2 z^2}} \quad (54)$$

showing that also at criticality the response function is time translational invariant.

After an integration by parts, the function $\chi_{\text{st}}(\tau, \infty)$ can be expressed in $d = 3$ in terms of the ${}_2F_1$ hypergeometric function as

$$\begin{aligned}
\chi_{\text{st}}(\tau, \infty) &= \frac{1}{(2\pi)^{3/2}} [\Lambda(1/2 + (\Lambda^2\tau + 1)^{-1/2}(4 + 1/3\gamma^2\tau^2)^{-1/2})] \\
&\quad - \frac{\gamma^2}{9} \tau^{3/2} {}_2F_1[1/2, 3/4, 7/4, -\gamma^2\tau^2/12] \quad (55)
\end{aligned}$$

while in $d = 2$, one gets

$$\begin{aligned}
\chi_{\text{st}}(\tau, \infty) &= \frac{1/(2\pi)}{\sqrt{1 + 1/(3A^2)}} \left[\log(\Lambda^2\tau + 1) + \log\left(1 + \sqrt{1 + 1/(3A^2)}\right) \right. \\
&\quad \left. - \log\left(1 + \sqrt{1 + 1/(3A^2)}\sqrt{1 + 1/3(\gamma\tau/2)^2} - \gamma\tau/(6A)\right) \right]. \quad (56)
\end{aligned}$$

In particular, for $\tau \ll \tau_s$, one has the limiting behaviour

$$\chi_{\text{st}}(\tau, \infty) = \begin{cases} \frac{1}{2\pi} \log(\Lambda^2\tau + 1) & d = 2 \\ \frac{M_0^2}{T_c(\gamma)} \left(1 - \frac{1}{\sqrt{\Lambda^2\tau + 1}}\right) & d = 3 \end{cases} \quad (57)$$

while

$$\chi_{\text{st}}(\tau, \infty) = \frac{M_0^2}{T_c(2\gamma)} \quad (58)$$

in the opposite limit $\tau \gg \tau_s$.

3. $T < T_c(\gamma)$. For $T < T_c(\gamma)$ the integrated response function is given by

$$\chi(t, t_w) = \frac{2}{(4\pi)^{d/2}} \int_0^\tau du \frac{(u + 1/(\Lambda^2))^{-d/2}}{\sqrt{4 + \gamma^2 u^2/3}} (1 - u/t)^{-(d+2)/4}. \quad (59)$$

The term $(1 - u/t)^{-(d+2)/4}$ can be expanded in powers of u/t and integrated term by term. It can be shown that the first contribution is dominant for $t_w \gg \tau_s$ and behaves as $\chi_{\text{st}}(\tau, \infty)$ in equation (54). Then in the limit $t_w \gg \tau_s$ the response function becomes TTI also in the low temperature phase.

It is interesting to study the behaviour of the first correction to this result, namely the next term appearing in the power series in u/t discussed above. The dependence on the two times of this quantity, which will be denoted as $\chi_{\text{ag}}(t, t_w)$, can be factorized as $\chi_{\text{ag}}(t, t_w) = t^{-1}b(\tau)$. Here $b(\tau)$ is a TTI function that, starting from zero at $\tau = 0$, saturates to a constant value for $\tau > \tau_s$. In the regime $\tau > \tau_s$, therefore, one has $\chi_{\text{ag}}(t, t_w) \sim t^{-1}$. In conclusion, taking into account also this first correction, it results that $\chi(t, t_w) \simeq \chi_{\text{st}}(\tau, \infty) + \chi_{\text{ag}}(t, t_w)$ and, recalling the behaviour of $\chi_{\text{ag}}(t, t_w)$, one concludes that for large t the second term is always negligible with respect to the first, as anticipated after equation (59).

A similar structure, with a stationary and an aging part, is also found in the case with $\gamma = 0$. However, in that case, one finds a similar result, namely $\chi_{\text{ag}}(t, t_w) = t^{-1}b(\tau)$ and the properties of $b(\tau)$ discussed above, only above the upper critical dimension $d_U = 4$. Instead, below d_U , $\chi_{\text{ag}}(t, t_w) = t_w^{-a}B(t/t_w)$, for $\tau/t_w > 1$, with $a = (d - 2)/2$. Note that this form implies that $\chi_{\text{ag}}(t, t_w)$ does not vanish for large t at the lower critical dimension $d = 2$. A similar pattern is also found for scalar systems [21].

When shear is applied the system behaves as in the case with $\gamma = 0$ and $d \geq d_U$. This is due to the fact that the mathematical structure of the solution of the model with applied flow in dimension d is similar to the structure of the solution without shear in an increased dimensionality $d + 2$. This can be checked, for instance, from the behaviour of the function $y(t)$ of equations (36) and (37) as compared to the form without shear [18]. This phenomenon raises the *effective* dimensionality at or above $d_U = 4$ for the cases $d = 2, 3$ considered in this paper.

7. Fluctuation–dissipation relation

In this section we will discuss the out of equilibrium generalization of the FDT, namely the relation between $D(t, t_w)$ and $\chi(t, t_w)$ in the large t_w regime.

1. $T > T_c(\gamma)$. Let us first recall that in the case without shear, for times t_w larger than the equilibration time, $R(t, t_w)$, $D(t, t_w)$ and $\chi(t, t_w)$ become TTI quantities $R(\tau)$, $D(\tau)$ and $\chi(\tau)$. Moreover, the equilibrium FDT holds, namely

$$TR(\tau) = -\frac{dD(\tau)}{d\tau} \quad (60)$$

which implies

$$T\chi(\tau) = D(\tau = 0) - D(\tau). \quad (61)$$

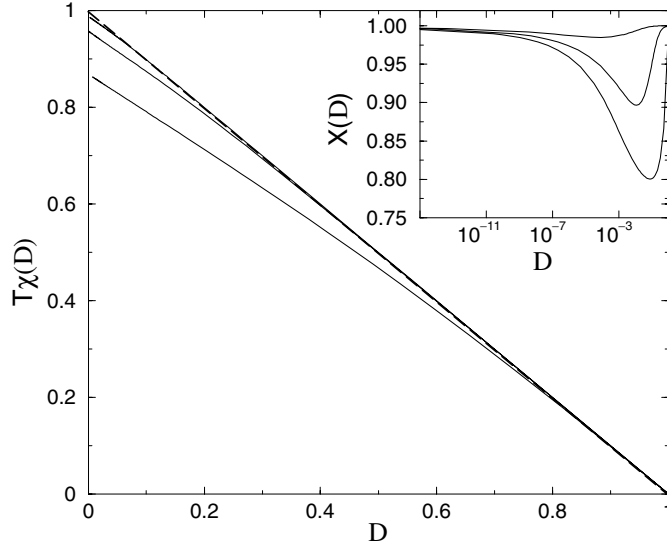


Figure 3. $T\chi(\tau)$ is plotted against $D(\tau)$ for a quench to a final temperature $T = (21/20)T_c(\gamma)$, with $t_w = 10^5$, $d = 3$, $\Lambda = 1$, $g = -r = 1$. Solid lines correspond to $\gamma = 10^{-2}$, 10^{-1} , 1 , from top to bottom. The dashed line, which is almost indistinguishable from the case $\gamma = 10^{-2}$, is the equilibrium behaviour $\gamma = 0$. The inset shows the fluctuation–dissipation ratio $X(D)$ defined in equation (63). Times are measured in units of Γ^{-1} , shear rates in units of Γ .

Note that equation (60) is equivalent to $T R_{\text{im}}(\omega) = \omega D_{\text{re}}(\omega)$, the usual relation between the dissipative part of the response and the spectral density [23]. Here $R_{\text{im}}(\omega)$ and $D_{\text{re}}(\omega)$ are the imaginary and real parts of the Fourier transform of $R(\tau)$ and $D(\tau)$.

The effect of the shear flow is to transfer energy to the system, and, also in the quench to a temperature above the critical point where TTI is obeyed, the equilibrium Gibbs state is never reached. Then the FDT does not hold and $D(t, t_w)$ and $\chi(t, t_w)$ do not satisfy an equation analogous to equation (61). The fact that $D(t, t_w) = D_{\text{st}}(\tau, \xi_{\perp})$ and $\chi(t, t_w) = \chi_{\text{st}}(\tau, \xi_{\perp})$ are TTI quantities implies that the dependence on the two times of $\chi(t, t_w)$ can still be accounted for through $D(t, t_w)$, yielding a non-trivial relation $\tilde{\chi}(D)$ between $\chi(t, t_w)$ and $D(t, t_w)$. The fluctuation–dissipation plot, namely the relation $\tilde{\chi}(D)$, is shown in figure 3.

The plot follows the straight line (61) for small values of τ , when $\tau \ll \tau_s$. Recalling the discussion of section 2 regarding local equilibrium one has that the system behaves as in equilibrium up to timescales of the order of the evolution time of the system which, in this case, is the flow timescale τ_s .

For larger values of τ (smaller D) $\tilde{\chi}(D)$ deviates from the equilibrium form (61) due to the effects of shear. In particular, for $T \gtrsim T_c(\gamma)$ its asymptotic ($\tau = \infty$) value is $T \tilde{\chi}(\tau, \xi_{\perp}) \approx M_0^2 T / T_c(2\gamma)$ (equation (58)) which is less than $M_0^2 - M^2 = M_0^2 T / T_c(\gamma)$, the value obtained from equation (61). From equation (58) it is readily seen that the relation between the response $\chi(t, t_w; 2\gamma)$ of a system subjected to a shear flow with rate 2γ , and the autocorrelation function $D(t, t_w; \gamma)$ of another system with shear rate γ , obeys equation (61) at large time ($\gamma\tau \gg 1$), namely

$$T\chi_{\text{st}}(\tau, \xi_{\perp}; 2\gamma) = D_{\text{st}}(0, \xi_{\perp}; \gamma) - D_{\text{st}}(\tau, \xi_{\perp}; \gamma). \quad (62)$$

This symmetry of the theory is independent of the model parameters and is therefore particularly suited for comparison with other models and for experimental checks. Actually,

the linearization of the equation of motion provided by the large- N model is expected to provide reliable results above the critical temperature and we expect relation (62) to be observed also in more realistic systems.

2. $T = T_c(\gamma)$. At $T = T_c(\gamma)$ the fluctuation–dissipation plot is the same as for $T \gtrsim T_c(\gamma)$. In the following we will focus on the fluctuation–dissipation ratio:

$$X(t, t') = T \frac{R(t, t')}{\partial D(t, t')/\partial t'} \quad (63)$$

and, in particular, on its limiting value

$$X_\infty = \lim_{t' \rightarrow \infty} \lim_{t \rightarrow \infty} X(t, t') \quad (64)$$

In systems without drive this quantity takes the value $X_\infty = 1$ for equilibrium systems while $X_\infty = 0$ in the low temperature phase of coarsening systems with a non-vanishing critical temperature [18, 21, 24]. For critical systems the value of X_∞ has been computed for the random walk and the Gaussian model where the non-trivial value $X_\infty = 1/2$ is found [25]. This quantity has also been computed for some coarsening systems at the lower critical dimension d_L , such as the X–Y model in $d = 2$ [25] and the Ising chain [26], where the same result $X_\infty = 1/2$ is recovered. However, concerning systems quenched to the critical temperature above d_L , the value of X_∞ turns out to possess a different value. Numerical simulations of the Ising model with heat bath dynamics in $d = 2$ indicate that $X_\infty \sim 0.26$ [27] and the exact solution [27] of the spherical model (which is equivalent to the large- N model) gives

$$X_\infty = \frac{d-2}{4} \quad 2 < d < 4 \quad (65)$$

$$X_\infty = \frac{1}{2} \quad d > 4. \quad (66)$$

In [27] it is proposed that this quantity is a novel universal quantity of non-equilibrium critical dynamics.

Under the action of a shear flow, equations (66) is obeyed for both $d = 2$ and $d = 3$, as we show below. Indeed, concentrating on the case $d = 3$ for clarity, taking $R(t, t')$ from equation (51) (with $y(t) = \text{const}$) in the asymptotic limit $t - t' \gg t'$ and evaluating $\partial D/\partial t'$ from equation (42), one obtains

$$X_\infty = \frac{1}{2}. \quad (67)$$

It can be shown that the same result $X_\infty = 1/2$ applies also in the case $d = 2$. Then, with respect to the case $\gamma = 0$ where equations (65) and (66) hold, the effect of the flow is to shift by two the *effective* dimensionality of the system, as already discussed in the previous section.

3. $T < T_c(\gamma)$. Summarizing the results of the previous sections, in the regime $t_w \gg \tau_s$, one has

$$D(t, t_w) = D_{\text{st}}(\tau, \infty) + D_{\text{ag}}(t/t_w) \quad (68)$$

$$\chi(t, t_w) = \chi_{\text{st}}(\tau, \infty). \quad (69)$$

Before discussing the behaviour of the flowing system it is useful to overview the behaviour without shear that is plotted in figure 4. In this case a structure such as (68) and (69) is also found.

In the short time separation regime (see section 5), or stationary regime, $D_{\text{st}}(\tau, \infty)$ decays from $D_{\text{st}}(0, \infty) = M_0^2 - M^2$ to $D_{\text{st}}(\tau \simeq t_w, \infty) \simeq 0$ while $D_{\text{ag}}(t/t_w) \simeq D_{\text{ag}}(1) = M^2$ remains constant. Therefore, in this regime $D(t, t_w)$ ranges from M_0^2 down to M^2 . $T \chi_{\text{st}}(\tau, \infty)$

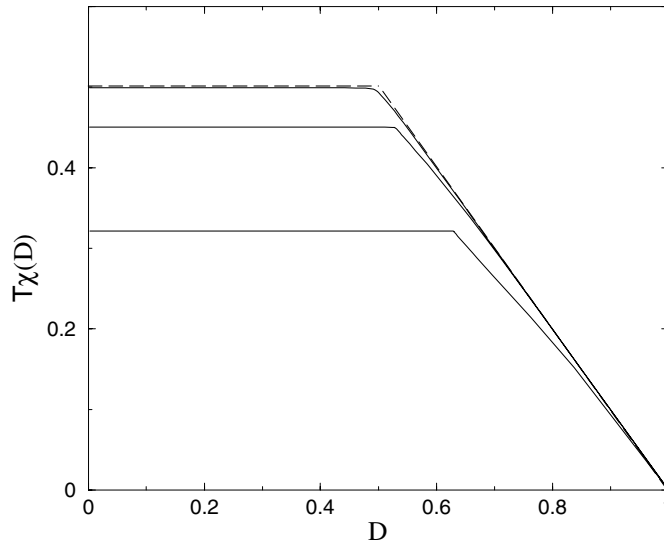


Figure 4. $T\chi(t, t_w)$ is plotted against $D(t, t_w)$ for a quench to a final temperature $T = T_c(\gamma)/2$, with $t_w = 10^5$, $d = 3$, $\Lambda = 1$, $-r = g = 1$. Solid lines correspond to $\gamma = 10^{-2}, 10^{-1}, 1$, from top to bottom. The dashed line is the case with $\gamma = 0$. Times are measured in units of Γ^{-1} , shear rates in units of Γ .

increases from zero to $T\chi_{st}(\tau \simeq t_w, \infty) \simeq M_0^2 - M^2$ and is related to $D_{st}(\tau, \infty)$ by the equilibrium FDT, equation (61). This gives rise to the straight line with negative slope on the right of figure 4. Note that this line is nothing other than the fluctuation–dissipation plot of a system in the equilibrium state above T_c , shifted by the amount M^2 along the horizontal axis.

In the large time separation sector, or aging regime, $D_{st}(\tau, \infty) \simeq 0$ whereas $D_{ag}(t/t_w)$ decays from $D_{ag}(1) = M^2$ to zero for large t/t_w . Hence the aging regime corresponds to the region $0 \leq D(t, t_w) \leq M^2$ of the fluctuation–dissipation plot (figure 4). The response function stays constant $T\chi(t, t_w) = M_0^2 - M^2$ because the aging part of this quantity is negligible for large t_w . This behaviour accounts for the flat part on the left side of figure 4. Given this structure it is also clear that the whole time dependence of the response can be absorbed through $D(t, t_w)$, namely $\chi(t, t_w) = \tilde{\chi}(D)$, as proposed in [2].

Next we turn to the case with shear. In this situation, given equations (68) and (69) and the behaviour of $D_{st}(\tau, \infty)$, $D_{ag}(t/t_w)$ and $\chi_{st}(\tau, \infty)$ obtained in sections 5 and 6 (equations (42), (44) and (54)), most of the considerations discussed for $\gamma = 0$ still apply. In particular, in the short time separation regime $D_{st}(\tau, \infty)$ decays from $D_{st}(0, \infty) = M_0^2 - M^2$ to zero while $D_{ag}(t/t_w) \simeq M^2$ stays constant. In this regime $D(t, t_w)$ ranges from M_0^2 down to M^2 and $T\chi_{st}(\tau, \infty)$ increases from zero to its limiting value $T\chi_{st}(\tau \simeq t_w, \infty)$. However, when shear is applied, the relation between $\chi_{st}(\tau, \infty)$ and $D_{st}(\tau, \infty)$ is not the equilibrium FDT (61) but, instead, the relation $\tilde{\chi}(D)$ of the case $T \gtrsim T_c(\gamma)$ discussed above (shifted by the amount M^2 along the D -axes, as for $\gamma = 0$). Therefore, as discussed already for $T \gtrsim T_c(\gamma)$ in the stationary regime, one has both a region $\tau < \tau_s$ where standard FDT (61) holds (on the right of figure 4), and a time domain $\tau_s < \tau < t_w$ where the relation $\tilde{\chi}(D)$ deviates from the linear behaviour (61). Finally, for $\tau > t_w$ (namely $D < M^2(\gamma)$), the aging regime is explored with a flat fluctuation–dissipation plot.

The pattern of violation of the FDT discussed so far for coarsening under shear flow can be compared with the behaviour of driven mean field models for glassy kinetics [3]. In both cases

the system without drive exhibits a phase transition at a critical temperature T_c characterized by an aging dynamics below T_c , and a qualitatively similar behaviour of the autocorrelation function. However, several differences occur between these systems when the drive is switched on, some of which are revealed by the fluctuation–dissipation relation. The most important difference is due to the fact that mean field glass models always attain asymptotically a stationary state, regardless of the temperature, while, as discussed in sections 3 and 5, the model considered in this paper exhibits an aging kinetics below a critical temperature even in the presence of shear. Furthermore, in glassy models, the fluctuation–dissipation relation $\chi(D)$ is a broken line for every temperature and the two slopes of these lines are associated with the existence of two well defined temperatures in the system, the bath temperature T and the effective temperature T_{eff} of the slow degrees of freedom. A similar behaviour is found in molecular dynamics simulations of binary Lennard-Jones mixture under shear flow [28]. On the other hand, at least in the stationary state observed above $T_c(\gamma)$, the only one that can be compared with the time translational invariant states of the glassy case, the model considered here shows a more complex relation $\chi(D)$, as can be clearly observed in the inset of figure 4. This suggests that, despite some qualitative similarities, coarsening systems under the action of an external drive behave quite differently from glassy systems and that the violation of the FDT is an efficient tool to detect it.

8. Conclusions

In this paper we have studied analytically the out of equilibrium kinetics of a solvable model for phase ordering systems subjected to a uniform shear flow. Besides the relevance of this subject as a first step for a better comprehension of sheared two-component fluids, which has recently been by itself a matter of intense theoretical and experimental interest, this model is particularly suited for investigating off-equilibrium systems under a general and wide perspective. Actually, the properties of non-equilibrium states are an important concern of modern statistical mechanics and many efforts have been devoted to widen our knowledge in this field. In particular, much interest has been paid to aging systems, namely systems which retain memory of the time t_w elapsed since they have been brought out of equilibrium, such as glasses or spin glasses. The aging properties are usually extracted from the behaviour of the autocorrelation function $D(t, t_w)$ which carries an explicit dependence on both times, up to the longest observation scales. On the other hand, the so-called driven systems, to which external energy is pumped from the outside, generally lose memory of the initial condition and, after a microscopic time, a time translational invariant state is entered, which, because of the external drive, is not an equilibrium one, in the Gibbs sense. It has been argued that an important piece of information on non-equilibrium states, both aging and stationary, is encoded into the linear response function to an external perturbation, and, in particular, in the so-called off-equilibrium generalization of the FDT, the relation between the response function and $D(t, t_w)$. Hopefully, this non-equilibrium linear response theory should bear the amount of relevant implications and the generality of the equilibrium case. However, nowadays, our understanding of this important issue is still incomplete and further investigations are necessary in order to deepen our insight.

The model we have studied in this paper is particularly suited for this analysis because it offers both a framework where exact calculations can be carried over and a rich phenomenology due to the presence of both stationary and aging off-equilibrium states. These two kinds of behaviour are separated by a phase transition at a critical temperature $T_c(\gamma)$ which increases with γ and gives a transition also in $d = 2$. We stress that this is a transition between non-equilibrium states. Nevertheless, in this model the existence of a non-equilibrium parameter

γ allows one to compare the properties of the transition with the well known equilibrium case with $\gamma = 0$. In doing that, one discovers that a wealth of similarities exist greatly helping the analysis of what goes on out of equilibrium. In particular, in the low temperature phase, where the aging behaviour coexists with the drive, the autocorrelation function can be split into a stationary and an aging part, along the same lines as for $\gamma = 0$. The former describes the correlation between fast degrees of freedom while the latter, which decays as a power law with a generalized Fisher–Huse exponent, is due to the slow variables responsible for the aging properties. This analysis allows a better comprehension of the violation of the FDT and the recognition that a mechanism very similar to that observed without shear is at work. This provides the basis for understanding why the fluctuation–dissipation plot of figure 4, with the flat part characteristic of coarsening systems, comes about also in sheared systems.

In aging systems without drive a definite progress in generalizing the FDT out of equilibrium has been achieved by realizing that certain fundamental properties of the system are encoded into the fluctuation–dissipation relation. A first step in this direction was made by interpreting $T_{\text{eff}} = T X^{-1}(C)$ as an effective temperature of the system, different from that of the thermal bath. Furthermore, it was also shown by Franz *et al* [29] that the relation between response and autocorrelation function bears information on the properties of the equilibrium state towards which the system is evolving. These advances qualify the fluctuation–dissipation relation as a fundamental one also far from equilibrium, providing an important tool in this difficult field. However, a generalization of these concepts to off-equilibrium driven systems is lacking, particularly for the low temperature phase of the model considered here, where aging coexists with the drive. Further studies are therefore needed to discover if and which properties of the non-equilibrium state are encrypted into their fluctuation–dissipation relation.

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Appendix A. The Laplace transform of $f(t)$

In this appendix we compute the Laplace transform of $f(t)$ defined as

$$f^L(s) = \int_0^\infty e^{-ts} f(t) dt \quad (70)$$

where $f(t)$ is given in equation (32). In doing so we will also obtain an explicit form for the quantity $P(x)$ defined in equation (18) due to the relation

$$P(x) = 2f^L(2x). \quad (71)$$

This relation can be easily derived by inserting expression (31) into equation (70) and comparing the result with definition (18).

1. $d = 2$. We first calculate the Laplace transform of the function $f(t)$ of equation (31) in $d = 2$. It can be written as

$$\begin{aligned} f^L(s) &= \int_0^\infty e^{-ts} f(t) dt = \frac{A}{4\pi} \int_0^\infty \frac{e^{-\frac{s}{\gamma}y}}{1+Ay} \left(4 - \frac{y^4 A^2}{(1+Ay)^2} + \frac{4}{3} \frac{y^3 A}{(1+Ay)} \right)^{-\frac{1}{2}} dy \\ &= \frac{e^{\frac{s}{\gamma A}}}{4\pi} \int_{\frac{s}{\gamma A}}^\infty dz \frac{e^{-z}}{z} \left(4 - \frac{(z\gamma A/s - 1)^4 s^2}{A^4 \gamma^2 z^2} + \frac{4}{3} \frac{(z\gamma A/s - 1)^3 s}{A^3 \gamma z} \right)^{-\frac{1}{2}} \end{aligned} \quad (72)$$

where the variable $z = s(1 + Ay)/(\gamma A)$ is introduced and A is defined in equation (21). The last integral can be split into two parts: the first is given by

$$\frac{e^{\frac{s}{\gamma A}}}{8\pi} \int_{\frac{s}{\gamma A}}^{\sqrt{12}s/\gamma} dz \frac{e^{-z}}{z} \frac{1}{\sqrt{1+h(z)}} \tag{73}$$

where the function

$$h(z) = \frac{(z\gamma A/s - 1)^3 s^2}{A^4 \gamma^2 z^2} \left(\frac{z\gamma A}{12s} + \frac{1}{4} \right) \tag{74}$$

can be shown to be less than one in the integration interval. Then the square root of equation (73) can be expanded as a power series and integrated term by term. The result is

$$\int_{\frac{s}{\gamma A}}^{\sqrt{12}s} dz \frac{e^{-z}}{z} \frac{1}{\sqrt{1+h(z)}} = \ln(\sqrt{12}A) + \sum_{n=1}^{\infty} \frac{r_n}{2n} - \sqrt{12} \frac{s}{\gamma} \sum_{n=0}^{\infty} \frac{r_n}{2n+1} + \mathcal{O}(s^2, A^{-1}) \tag{75}$$

with $r_n = (-1)^n (2n - 1)! / (2n)!!$. The other contribution to $f^L(s)$, neglecting all terms of order $\mathcal{O}(s^2, A^{-1})$, is given by

$$\frac{e^{\frac{s}{\gamma A}}}{8\pi} \int_{\frac{\sqrt{12}s}{\gamma}}^{\infty} dz \frac{e^{-z}}{z} \frac{1}{\sqrt{1 + \frac{z^2 \gamma^2}{12s^2}}}. \tag{76}$$

Proceeding along the same lines as for the previous integral (73) and adding the result to expression (75) one finds

$$f^L(s) = \frac{1}{8\pi} \left(\ln(\sqrt{12}A) + 1 + D_0 + \sqrt{12}D_1 \frac{s}{\gamma} + \sum_{n=0}^{\infty} \frac{12^{n+1/2} s^{2n+1}}{\gamma^{2n+1}} \left(\prod_{k=0}^{2n} \frac{1}{k+1} \right) \ln \frac{\sqrt{12}s}{\gamma} \right) + \mathcal{O}(s^2, A^{-1}) \tag{77}$$

where $D_0 = \sum_{n=1}^{\infty} r_n \frac{(4n+1)}{2n(2n+1)} = \ln 2 - 1 \approx -0.3068$, $D_1 = \mathcal{C} - 2 - D_0 \approx -1.7296$ with $\mathcal{C} \approx 0.577 216$ being Euler’s constant. Equation (77) can be written more compactly as

$$f^L(s) \approx \frac{M_0^2}{2T_c(\gamma)} - A_2 s + B_2 s \log s \tag{78}$$

where only the first term in the series of equation (77) is retained and

$$f^L(0) = \frac{M_0^2}{2T_c(\gamma)} = \frac{1}{8\pi} (\ln(\sqrt{12}A) + \ln 2) \tag{79}$$

$$A_2 = -\frac{\sqrt{12}}{8\pi} \gamma^{-1} (D_1 - \ln(\sqrt{12}/\gamma)) \quad B_2 = \frac{\sqrt{12}}{8\pi} \gamma^{-1}. \tag{80}$$

2. $d = 3$. Similar to the case $d = 2$, starting from equation (32), $f^L(s)$ can be written as

$$f_3^L(s) = \frac{\sqrt{\gamma} A^{3/2}}{4(2\pi)^{3/2}} \int_0^{\infty} \frac{e^{-\frac{s}{\gamma}y}}{(1 + Ay)^{3/2}} \left(4 - \frac{y^4 A^2}{(1 + Ay)^2} + \frac{4}{3} \frac{y^3 A}{(1 + Ay)} \right)^{-\frac{1}{2}} dy$$

$$= \frac{e^{\frac{s}{\gamma A}}}{4(2\pi)^{3/2}} \sqrt{s} \int_{\frac{s}{\gamma A}}^{\infty} dz \frac{e^{-z}}{z^{3/2}} \left(4 - \frac{(z\gamma A/s - 1)^4 s^2}{A^4 \gamma^2 z^2} + \frac{4}{3} \frac{(z\gamma A/s - 1)^3 s}{A^3 \gamma z} \right)^{-\frac{1}{2}}. \tag{81}$$

The last expression can again be calculated by splitting the integration interval. A first contribution comes from the term

$$\begin{aligned} \frac{\sqrt{s}}{2} \int_{\frac{s}{\gamma A}}^{\sqrt{12s/\gamma}} dz \frac{e^{-z}}{z^{3/2} \sqrt{1+h(z)}} &= \sqrt{A\gamma} + \frac{\sqrt{\gamma}}{2(12)^{1/4}} \sum_{n=0}^{\infty} \frac{r_n}{2n-1/2} \\ &- \frac{12^{1/4}}{2} \frac{s}{\sqrt{\gamma}} \sum_{n=0}^{\infty} \frac{r_n}{2n+1/2} + \mathcal{O}(s^2, A^{-1/2}). \end{aligned} \tag{82}$$

Adding this contribution to the other term

$$\begin{aligned} \frac{\sqrt{s}}{2} \int_{\frac{\sqrt{12s}}{\gamma}}^{\infty} dz \frac{e^{-z}}{z^{3/2} \sqrt{1+\frac{z^2\gamma^2}{12s^2}}} &= \frac{\sqrt{\gamma}}{2(12)^{1/4}} \sum_{n=0}^{\infty} \frac{r_n}{2n+3/2} - 12^{1/4} \frac{s}{2\sqrt{\gamma}} \sum_{n=0}^{\infty} \frac{r_n}{2n+1/2} \\ &+ \frac{\Gamma(1/2)}{2} \sum_{n=0}^{\infty} \left(\prod_{k=0}^{2n+1} \frac{2}{2k+1} \right) s^{2n+3/2} \frac{12^{n+1/2}}{\gamma^{2n+1}} + \mathcal{O}(s^2, A^{-1/2}) \end{aligned} \tag{83}$$

one obtains

$$\begin{aligned} f^L(s) &= \frac{1}{4(2\pi)^{3/2}} \left[\sqrt{\gamma A} + \frac{\sqrt{\gamma}}{12^{1/4}} T_0 - 12^{1/4} \frac{s}{\sqrt{\gamma}} T_1 \right. \\ &\left. + \frac{\Gamma(1/2)}{2} \sum_{n=0}^{\infty} \left(\prod_{k=0}^{2n+1} \frac{2}{2k+1} \right) s^{2n+3/2} \frac{12^{n+1/2}}{\gamma^{2n+1}} \right] + \mathcal{O}(s^2, A^{-1/2}) \end{aligned} \tag{84}$$

where $T_0 = \sum_{n=0}^{\infty} r_n \frac{12^{n+1/2}}{(2n+3/2)(2n-1/2)} \approx -0.847$, $T_1 = \sum_{n=0}^{\infty} r_n \frac{1}{2n+1/2} \approx 1.8541$. The behaviour for small s is only relevant for the asymptotic dynamics, as discussed below equation (94). This is given by

$$f^L(s) \approx \frac{M_0^2}{2T_c(\gamma)} - A_3 s + B_3 s^{3/2} \tag{85}$$

with

$$f^L(0) = \frac{M_0^2}{2T_c(\gamma)} = \frac{1}{4(2\pi)^{3/2}} \left(\sqrt{\gamma A} + \frac{\sqrt{\gamma}}{12^{1/4}} T_0 \right) \tag{86}$$

$$A_3 = \frac{12^{1/4}}{4(2\pi)^{3/2}} \frac{T_1}{\sqrt{\gamma}} \quad B_3 = \frac{\sqrt{\pi} 12}{6(2\pi)^{3/2}} \gamma^{-1}. \tag{87}$$

It can be shown that the coefficients T_0, T_1 can also be written in terms of the Γ -function as $T_0 = -\frac{\Gamma^2(3/4)}{\sqrt{\pi}}, T_1 = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}$.

Appendix B. The function $y(t)$

We want to compute $y(t)$, the inverse Laplace transformation of $y^L(s)$ of equation (35), defined by

$$y(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds y^L(s) e^{st} \tag{88}$$

where σ has to be chosen in such a way that all the poles of $y^L(s)$ are on the left of the path of integration.

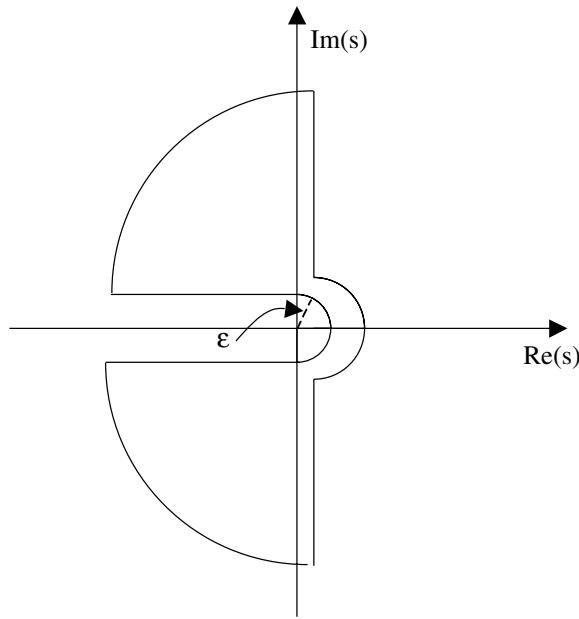


Figure 5. The closed contour adopted to compute the integral (88).

We first study the pole structure of $y^L(s)$ in the half-plane $x > 0$ with $s = x + iy$. The real part $R(x, y)$ of the denominator of $y^L(s)$ is given by

$$R(x, y) = x - 2r - 4T \operatorname{Re}(f^L(x, y)) \tag{89}$$

where

$$\operatorname{Re}(f^L(x, y)) = \int_0^\infty dt f(t) e^{-xt} \cos(yt). \tag{90}$$

Given that $f(t) \geq 0$ for any time (see equation (32)), for non-negative values of x , $\operatorname{Re}(f^L(x, y))$ is a monotone decreasing function of both x and y . This implies, from equation (89), that $R(x, y)$ is a monotone increasing function in the half-plane considered. As a consequence, $R(x, y)$ remains positive for all $x \geq 0$ if $R(0, 0) > 0$. Since from the definition of $T_c(\gamma)$ given in equation (23) one has

$$R(0, 0) > 0 \quad T < T_c(\gamma) \tag{91}$$

$$R(0, 0) = 0 \quad T = T_c(\gamma) \tag{92}$$

$$R(0, 0) < 0 \quad T > T_c(\gamma) \tag{93}$$

for $T \leq T_c(\gamma)$, $R(x, y)$ is always positive for $x > 0$ and $y^L(s)$ has no poles in the complex positive half-plane. In particular, at $T = T_c(\gamma)$ there is a pole at the origin. On the other hand, for $T > T_c(\gamma)$, the monotonic behaviour of $R(x, y)$ and equation (93) imply the existence of a positive p such that $R(x, y) > R(p, 0) = 0$ for any $x > p$. Since the imaginary part of the denominator of $y^L(s)$ is zero everywhere on the real axis, one can conclude that the largest pole is located on the real axis at p and, in the Bromwich contour (the vertical integration path of figure 5), $\sigma > p$.

We can now calculate explicitly the integral of equation (88) in the different temperature regimes.

1. $T < T_c(\gamma)$. In this case there are no poles in the positive complex plane and the Bromwich contour can coincide with the imaginary axis. The theorem of residues can be applied choosing a closed contour in the negative complex half-plane. Due to the non-analytical part of $f^L(s)$, there is a branch cut in the complex plane along the negative real axis. Then we choose the contour Θ of figure 5 and the only non-vanishing contributions come from the Bromwich contour itself and from the integral along both sides of the branch cut. Then, one has

$$\int_{\Theta} ds y^L(s) e^{st} = 2i\pi y(t) + \int_0^{\infty} dx e^{-xt} (y^L(s)|_{s=x e^{i\pi}} - y^L(s)|_{s=x e^{-i\pi}}). \quad (94)$$

Since we are interested in the large time behaviour of $y(t)$, we can compute the above integrals taking only the small s behaviour of $f^L(s)$ given in equations (78) and (85). We obtain

$$\int_{\Theta} ds y^L(s) e^{st} = 2i\pi y(t) - 2\pi i \frac{B_2}{M^4} (T + \Delta_0 M_0^2) t^{-2} \quad d = 2 \quad (95)$$

$$\int_{\Theta} ds y^L(s) e^{st} = 2i\pi y(t) - 2\sqrt{\pi} i \frac{3}{4} \frac{B_3}{M^4} (T + \Delta_0 M_0^2) t^{-5/2} \quad d = 3. \quad (96)$$

The above expressions are equal to the sum of the residues of all eventual poles inside Θ . In the negative complex plane these poles can only give contributions exponentially decreasing with time which are asymptotically negligible with respect to the power law behaviour of equations (95) and (96). Henceforth we obtain expressions (36) and (37) in $d = 2$ and $d = 3$, respectively.

2. $T = T_c(\gamma)$. The same contour Θ of the $T < T_c(\gamma)$ case can be chosen since the pole located at the origin is outside Θ . The evaluation of the integrals proceeds differently in $d = 2$ and $d = 3$. In $d = 2$ the computation strictly follows the one presented for $T < T_c(\gamma)$. The non-vanishing contributions on Θ still come from the Bromwich contour and the integrals around the cut. One obtains

$$\int_{\Theta} ds y^L(s) e^{st} = 2i\pi y(t) - 2\pi i \frac{1}{2B_2 T_c(\gamma)} \left(\frac{1}{2} + \frac{\Delta_0 M_0^2}{2T_c(\gamma)} \right) \frac{1}{\log t} \quad (97)$$

which gives, finally, the result of equation (36). In $d = 3$ the contribution coming from the small half-circle around the origin gives a constant term. Since, as before, the integral along both sides of the cut has a decreasing power law behaviour, asymptotically, the contribution from the half-circle around the origin is the dominant one and

$$\int_{\Theta} ds y^L(s) e^{st} = 2i\pi y(t) - \pi i \frac{1 + M_0^2 \Delta_0 / T_c(\gamma)}{1 + 4T_c(\gamma) A_3} \quad (98)$$

resulting in equation (37).

3. $T \gtrsim T_c(\gamma)$. In the case of a temperature $T = T_c(\gamma) + \delta T$ with $\delta T / T_c(\gamma) \ll 1$ the preliminary discussion about $R(x, y)$ assures the presence of a pole on the positive real axis close to the origin. The location $(p, 0)$ of this pole is given by the solution of the equation

$$R(p, 0) = 0 \quad (99)$$

that, for $p \ll 1$, as $\delta T / T_c(\gamma) \ll 1$ implies, reads

$$p \log(p) = M_0^2 \delta T / (T_c(\gamma) 2B_2 T) \quad d = 2 \quad (100)$$

$$p(1/2 + 2T A_3) = M_0^2 \delta T / T_c(\gamma) \quad d = 3. \quad (101)$$

At this point a similar procedure as in the case $T = T_c(\gamma)$ can be followed giving the asymptotic behaviour for $y(t)$ of equations (36) and (37) in $d = 2$ and $d = 3$, respectively.

Appendix C. Asymptotic behaviour of the zero mode of $C(\vec{k}, t)$ for $T < T_c(\gamma)$

The value $C(\vec{k} = 0, t_0)$ can be obtained from equation (28) and reads

$$C(\vec{k} = 0, t_0) = \frac{\Delta_0}{y(t_0)} + \frac{2T}{y(t_0)} \int_0^{t_0} y(z) dz. \quad (102)$$

The integral appearing in the above equation can be split into two terms:

$$\int_0^{t_0} y(z) dz = \int_0^\infty y(z) dz - \int_{t_0}^\infty y(z) dz. \quad (103)$$

From equation (35) one has

$$\int_0^\infty y(z) dz = y^L(0) = \frac{1}{2M^2} + \frac{M_0^2}{M^2} \frac{\Delta_0}{2T_c(\gamma)} \quad (104)$$

where we have used equations (79) and (86), while the asymptotic behaviour of $y(t)$ (equations (36) and (37)) gives

$$\int_{t_0}^\infty y(z) dz = \frac{1}{(8\pi)^{d/2}} \frac{\sqrt{12} T + \Delta_0 M_0^2}{\gamma} \frac{2}{M^4} t_0^{-d/2}. \quad (105)$$

Putting together equations (102), (104) and (105) one finally obtains

$$C(\vec{k} = 0, t_0) = \frac{1}{y(t_0)} \left(M^2 - \frac{4T}{(8\pi)^{d/2}} \frac{\sqrt{12}}{d\gamma} t_0^{-d/2} \right) \frac{T + \Delta_0 M_0^2}{M^4}. \quad (106)$$

Note that the dependence on the initial condition, namely the parameter Δ_0 , drops out for $t \gg \tau_s$ in equation (106), by virtue of expressions (36) and (37) of $y(t_0)$, because the scaling regime is entered and the memory of the initial state is lost.

Appendix D. The autocorrelation function

The evolution equation (40) for the autocorrelation function $D(t, t')$ can be integrated starting from a generic time t_0 chosen arbitrarily between the time of the quench ($t = 0$) and the observation time t . The formal solution for $D(t, t')$ then reads

$$D(t, t') = D_T(t, t') + D_0(t, t') \quad (107)$$

where

$$D_T(t, t') = \frac{2T}{\sqrt{y(t')y(t)}} \int \frac{d^d k}{(2\pi)^d} e^{-(k^2 + d(\gamma)/2)/\Lambda^2} \int_{t_0}^{t'} dz e^{-2 \int_0^{t-z} \mathcal{K}^2(u) du} e^{\int_0^{t'-z} \mathcal{K}^2(u) du} y(z) \quad (108)$$

and

$$D_0(t, t') = \frac{y(t_0)}{\sqrt{y(t')y(t)}} \int \frac{d^d k}{(2\pi)^d} C(\mathcal{K}(t - t_0), t_0) e^{-(k^2 + d(\gamma)/2)/\Lambda^2} e^{-2 \int_0^{t-t_0} \mathcal{K}^2(u) du} e^{\int_0^{t'} \mathcal{K}^2(u) du} \quad (109)$$

with $d(\gamma) = \mathcal{K}^2(t - t') - k^2$.

We will take the time t_0 large enough ($t_0 \gg \tau_s$) that $y(t)$ is given by equations (36) and (37). Furthermore, it can be shown that, due to the condition (21), the function $d(\gamma)$ can be neglected in equations (109) and (108).

We will first consider the function $D_T(t, t')$. Carrying out the integral one obtains

$$D_T(t, t') = \frac{4T}{(8\pi)^{d/2} \sqrt{y(t)y(t')}} \times \int_{t_0}^{t'} y(z)(t_m - z + \tau_M)^{-d/2} \frac{1}{\sqrt{4 - \gamma^2 \frac{((t-z)^2 - \tau^2/2)^2}{(t_m - z + \tau_M)^2} + \frac{4}{3} \gamma^2 \frac{((t-z)^3 - \tau^3/2)}{(t_m - z + \tau_M)}}} dz \quad (110)$$

with $\tau = t - t'$ and $t_m = (t + t')/2$. The role of the cut-off Λ is crucial in the factor $(t_m - z + \tau_M)^{-d/2}$ in order to avoid divergences at $z = t_m$; however, for large $t_m > \tau_s$, one finds that it can be neglected in the two terms under the square root. Then, introducing the variable $u = 1 - z/t_m$, one has

$$D_T(t, t') = \frac{4T}{(8\pi)^{d/2}} \frac{t_m^{-d/2}}{\gamma} \int_{\tau/(2t_m)}^{1-t_0/t_m} \rho(u, \tau, t_m) \Theta(u, \tau, t_m) du \quad (111)$$

where

$$\rho(u, \tau, t_m) = \frac{(u + 1/(2\Lambda^2 t_m))^{-d/2}}{\sqrt{\frac{4}{\gamma^2 t_m^2} + \frac{1}{3} u^2 + \frac{1}{2} \left(\frac{\tau}{t_m}\right)^2 \left(1 - \frac{(\tau/t_m)^2}{8u^2}\right)}} \quad (112)$$

and

$$\Theta(u, \tau, t_m) = \begin{cases} e^{-2\xi_{\perp}^{-2} t_m u} & \text{for } T > T_c(\gamma) \\ 1 & \text{for } T = T_c(\gamma) \quad d = 3 \\ \frac{\sqrt{\log(t_m + \tau/2) \log(t_m - \tau/2)}}{\log[t_m(1 - u)]} & \text{for } T = T_c(\gamma) \quad d = 2 \\ \left(\frac{1 - u}{\sqrt{1 - (\tau/2t_m)^2}}\right)^{-\frac{d+2}{2}} & \text{for } T < T_c(\gamma). \end{cases} \quad (113)$$

Turning now to the function $D_0(t, t')$, in the limit $t'/t_0 \gg 1$, due to the presence of the exponential factors in equation (109), $D_0(t, t')$ can be evaluated by approximating the one-time correlation function $C(\mathcal{K}(t - t_0), t_0)$ at the time t_0 entering equation (108), whose expression is given in equation (28), with its value at $\vec{k} = 0$. Then one has

$$D_0(t, t') = \frac{C(\vec{k} = 0, t_0) y(t_0)}{\sqrt{y(t)y(t')}} (t - \tau/2 + 1/2\Lambda^2)^{-d/2} \times \frac{2}{\sqrt{4 - \gamma^2 \frac{(t^2 - \tau^2/2)^2}{(t - \tau/2 + 1/2\Lambda^2)^2} + \frac{4}{3} \gamma^2 \frac{(t^3 - \tau^3/2)}{(t - \tau/2 + 1/2\Lambda^2)}}} \quad (114)$$

From now on one must distinguish different temperature ranges.

1. $T > T_c(\gamma)$. Starting with $D_T(t, t')$, from equations (111)–(113), one has

$$D_T(t, t') = \frac{4T}{(8\pi)^{d/2}} \int_{\tau/2}^{t_m - t_0} e^{-2\xi_{\perp}^{-2} y} (y + 1/2\Lambda^2)^{-d/2} \frac{1}{\sqrt{4 + \frac{1}{3} \gamma^2 y^2 + \frac{\gamma^2}{2} \tau^2 \left(1 - \frac{\tau^2}{8y^2}\right)}} dy. \quad (115)$$

In the limit $t_m \rightarrow \infty$, $D_T(t_m, \tau)$ becomes a TTI quantity

$$D_{st}(\tau, \xi_{\perp}) = \frac{4T}{(8\pi)^{d/2}} \int_{\tau/2}^{\infty} e^{-2\xi_{\perp}^{-2} y} (y + 1/2\Lambda^2)^{-d/2} \frac{1}{\sqrt{4 + \frac{1}{3} \gamma^2 y^2 + \frac{\gamma^2}{2} \tau^2 \left(1 - \frac{\tau^2}{8y^2}\right)}} dy. \quad (116)$$

With respect to $D_0(t, t')$, by inserting the asymptotic behaviour of $y(t)$ (equations (36) and (37)) in equation (114), one immediately obtains

$$D_0(t, t') \sim e^{-2\xi_{\perp}^{-2}(t+t')/2}. \quad (117)$$

From expressions (116) and (117) it is clear that, in the limit $t' \gg \xi_{\perp}^2$, $D_0(t, t')$ is negligible with respect to $D_T(t, t')$ so that

$$D(t, t') \simeq D_{\text{st}}(\tau, \xi_{\perp}) \quad (118)$$

and $D_{\text{st}}(\tau, \xi_{\perp})$ is given in equation (116).

2. $T = T_c(\gamma)$. In this case the same results (115) and (116) are found for $D_T(t, t')$ but with $\xi_{\perp}^{-1} = 0$. In particular, the large t_m behaviour of $D(t, t')$ is given by

$$D_{\text{st}}(\tau, \infty) = \frac{4T}{(8\pi)^{d/2}} \int_{\tau/2}^{\infty} (y + 1/2\Lambda^2)^{-d/2} \frac{1}{\sqrt{4 + \frac{1}{3}\gamma^2 y^2 + \frac{\gamma^2}{2}\tau^2(1 - \frac{\tau^2}{8y^2})}} dy \quad (119)$$

with corrections of order $t_m^{-d/2}$.

Regarding $D_0(t, t')$, from equations (36) and (37) one has

$$D_0(t, t') \sim \left(\frac{t+t'}{2}\right)^{-(d+2)/2} \quad (120)$$

with logarithmic corrections in $d = 2$.

Comparing equation (119) with equation (120) one can show that again, in the limit $t' \gg \tau_s$, $D_0(t, t')$ is negligible with respect to $D_T(t, t')$, as for $T > T_c(\gamma)$. Therefore

$$D(t, t') \simeq D_{\text{st}}(\tau, \infty). \quad (121)$$

The expression for $D_{\text{st}}(\tau, \infty)$ is equation (119).

3. $T < T_c(\gamma)$. For $T < T_c(\gamma)$ we study the behaviour of $D_T(t, t')$ separately in the temporal regimes $\tau/t_m \ll 1$ (*quasi-stationary regime*) and $\tau/t_m \gg 2/3$ ($\tau/t' \gg 1$) (*aging regime*). In the quasi-stationary regime, the function $\rho(u, \tau, t_m)$ of equation (112) is of order $t_m^{(d+2)/2}$ for $u \ll 1$ and decreases to zero as $\rho(u, \tau, t_m) \sim u^{-(d+2)/2}$ when $u > 1/\gamma t_m$. In the same regime, $\Theta \simeq 1$ when $u \ll 1$ while it diverges like $(1-u)^{-(d+2)/2}$ for $u \rightarrow 1 - t_0/t_m$. In the limits $\gamma t_m \rightarrow \infty$ and $t_0/t_m \rightarrow 0$, the leading contributions to the integral of equation (111) coming from the singularities of the functions $\rho(u, \tau, t_m)$ for $u \rightarrow 0$ and of $\Theta(u, \tau, t_m)$ for $u \rightarrow 1 - t_0/t_m$ give

$$\begin{aligned} D_T(t_m, \tau) &= \frac{4T}{(8\pi)^{d/2}} \frac{t_m^{-d/2}}{\gamma} \left[\int_{\tau/(2t_m)}^{\infty} du \rho(u, \tau, t_m) + \sqrt{3} \int_0^{1-t_0/(2t_m)} du \Theta(u, \tau, t_m) \right] + \mathcal{O}(t_m^{-d/2}) \\ &= D_{\text{st}}(\tau, \infty) + \frac{4\sqrt{12}T}{(8\pi)^{d/2}d\gamma} t_0^{-d/2} + \mathcal{O}(t_m^{-d/2}). \end{aligned} \quad (122)$$

Similar considerations applied to the aging regime imply

$$\begin{aligned} D_T(t_m, \tau) &= \frac{4T}{(8\pi)^{d/2}} \frac{t_m^{-d/2}}{\gamma} \frac{\sqrt{3}}{\sqrt{1 + \frac{3}{2}\left(\frac{\tau}{t_m}\right)^2 \left(1 - \frac{1}{8}\left(\frac{\tau}{t_m}\right)^2\right)}} \int_0^{1-t_0/(2t_m)} du \Theta(u, \tau, t_m) + \mathcal{O}(t_m^{-d/2}) \\ &= \frac{4T}{(8\pi)^{d/2}} \frac{\sqrt{12}}{d\gamma} t_0^{-d/2} \left(\frac{t}{t'}\right)^{-\frac{d+2}{4}} (1+t'/t)^{-\frac{d+2}{2}} \frac{2^{\frac{d+2}{2}}}{\sqrt{4\frac{2-(1-t'/t)^3}{1+t'/t} - 3\frac{(2-(1-t'/t)^2)^2}{(1+t'/t)^2}}} \\ &\quad + \mathcal{O}(t_m^{-d/2}). \end{aligned} \quad (123)$$

For $D_0(t, t')$ one finds

$$D_0(t, t') = \left(M^2 - \frac{4T}{(8\pi)^{d/2}} \frac{\sqrt{12}}{d\gamma} t_0^{-d/2} \right) \left(\frac{t}{t'} \right)^{-\frac{d+2}{4}} (1+t'/t)^{-\frac{d+2}{2}} \\ \times \frac{2^{\frac{d+2}{2}}}{\sqrt{4 \frac{2-(1-t'/t)^3}{1+t'/t} - 3 \frac{(2-(1-t'/t)^2)^2}{(1+t'/t)^2}}} \quad (124)$$

where the expression of $C(\vec{k} = 0, t_0)$ obtained in appendix C has been used and the limit $t' \gg \tau_s$ is assumed.

Summarizing, summing up $D_0(t, t')$ (equation (124)) with $D_T(t, t')$ (equation (122)), in the quasi-stationary regime one obtains

$$D(t, t') \simeq D_{\text{st}}(\tau, \infty) + M^2 \quad (125)$$

with $D_{\text{st}}(\tau, \infty)$ given in equation (119) while, in the aging regime, from equations (124) and (123) one has

$$D(t, t') = D_{\text{ag}}(t/t') = M^2 \left(\frac{t}{t'} \right)^{-\frac{d+2}{4}} (1+t'/t)^{-\frac{d+2}{2}} \frac{2^{\frac{d+2}{2}}}{\sqrt{4 \frac{2-(1-t'/t)^3}{1+t'/t} - 3 \frac{(2-(1-t'/t)^2)^2}{(1+t'/t)^2}}}. \quad (126)$$

thus defining the quantity $D_{\text{ag}}(t/t')$.

References

- [1] Hohenberg P C and Shraiman B I 1989 *Physica D* **37** 109
- [2] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173
Cugliandolo L F and Kurchan J 1995 *Phil. Mag.* **71** 501
Cugliandolo L F and Kurchan J 1994 *J. Phys. A: Math. Gen.* **27** 5749
- [3] Cugliandolo L F, Kurchan J and Peliti L 1997 *Phys. Rev. E* **55** 3898
Berthier L, Barrat J-L and Kurchan J 2000 *Phys. Rev. E* **61** 5464
- [4] Fielding S and Sollich P 2002 *Phys. Rev. Lett.* **88** 050603 (*Preprint cond-mat/0209645*)
- [5] Schmittmann B and Zia R K P 1995 Statistical mechanics of driven diffusive systems *Phase Transitions and Critical Phenomena* vol 17 ed C Domb and J L Lebowitz (New York: Academic)
- [6] Gallavotti G and Cohen E G D 1995 *Phys. Rev. Lett.* **74** 2694
Gallavotti G and Cohen E G D 1995 *J. Stat. Phys.* **80** 931
- [7] See, e.g., Larson R G 1999 *The Structure and Rheology of Complex Fluids* (New York: Oxford University Press)
- [8] For a review, see Onuki A 1997 *J. Phys.: Condens. Matter* **9** 6119
- [9] Corberi F, Gonnella G and Lamura A 1999 *Phys. Rev. Lett.* **83** 4057
- [10] Corberi F, Gonnella G and Lamura A 1998 *Phys. Rev. Lett.* **81** 3852
- [11] Corberi F, Gonnella G and Lamura A 2002 *Phys. Rev. E* **66** 016114
- [12] Bray A J 1994 *Adv. Phys.* **43** 357
- [13] Bray A J and Cavagna A 2000 *J. Phys. A: Math. Gen.* **33** L305
- [14] Coniglio A, Ruggiero P and Zannetti M 1994 *Phys. Rev. E* **50** 1046
- [15] Onuki A and Kawasaki K 1979 *Ann. Phys., NY* **121** 456
- [16] Kreuzer H J 1981 *Nonequilibrium Thermodynamics and its Statistical Foundations* ed H Frölich, P B Hirsch and N F Mott (Oxford: Oxford University Press)
- [17] Gonnella G and Pellicoro M 2000 *J. Phys. A: Math. Gen.* **33** 7043
Chan C and Lin L 1990 *Europhys. Lett.* **11** 13
- [18] Corberi F, Lippiello E and Zannetti M 2002 *Phys. Rev. E* **65** 046136
- [19] Mazenko G F, Valls O T and Zannetti M 1988 *Phys. Rev. B* **38** 520
Franz S and Virasoro M A 2000 *J. Phys. A: Math. Gen.* **33** 891
- [20] Corberi F, Gonnella G, Lippiello E and Zannetti M 2002 *Europhys. Lett.* **60** 425
- [21] Corberi F, Lippiello E and Zannetti M 2001 *Phys. Rev. E* **63** 061506
Corberi F, Lippiello E and Zannetti M 2001 *Eur. Phys. J. B* **24** 359

- [22] Fisher K H and Hertz J A 1989 *Spin Glasses* (Cambridge: Cambridge University Press)
- [23] de Groot S R and Mazur P 1984 *Non-Equilibrium Thermodynamics* (New York: Dover)
- [24] Barrat A 1998 *Phys. Rev. E* **57** 3629
- [25] Cugliandolo L F, Kurchan J and Parisi G 1994 *J. Physique I* **4** 1641
- [26] Godreche C and Luck J M 2000 *J. Phys. A: Math. Gen.* **33** 1151
Lippiello E and Zannetti M 2000 *Phys. Rev. E* **61** 3369
- [27] Godreche C and Luck J M 2000 *J. Phys A: Math. Gen.* **33** 9141
- [28] Barrat J-L and Berthier L 2001 *Phys. Rev. E* **63** 012503 (*Preprint cond-mat/0111312*)
- [29] Franz S, Mezard M, Parisi G and Peliti L 1998 *Phys. Rev. Lett.* **81** 1758
Franz S, Mezard M, Parisi G and Peliti L 1998 *J. Stat. Phys.* **97** 459